Secular interactions between inclined planets and a gaseous disk

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ABSTRACT

In a planetary system, a secular particle resonance occurs at a location where the precession rate of a test particle (e.g. an asteroid) matches the frequency of one of the precessional modes of the planetary system. We investigate the secular interactions of a system of mutually inclined planets with a gaseous protostellar disk that may contain a secular nodal particle resonance. We determine the normal modes of some mutually inclined planet-disk systems. The planets and disk interact gravitationally, and the disk is internally subject to the effects of gas pressure, self-gravity, and turbulent viscosity. The behavior of the disk at a secular resonance is radically different from that of a particle, owing mainly to the effects of gas pressure. The resonance is typically broadened by gas pressure to the extent that global effects, including large-scale warps, dominate. The standard resonant torque formula is invalid in this regime. Secular interactions cause a decay of the inclination at a rate that depends on the disk properties, including its mass, turbulent viscosity, and sound speed. For a Jupiter-mass planet embedded within a minimum-mass solar nebula having typical parameters, dissipation within the disk is sufficient to stabilize the system against tilt growth caused by mean-motion resonances.

Subject headings: accretion, accretion disks — celestial mechanics — hydrodynamics — planets and satellites: general — solar system: general — waves

1. Introduction

The interaction between a young planetary system and its protoplanetary disk likely plays an important role in determining the orbital properties of the planets. Such interactions may be important for understanding the observed orbital properties of extra-solar planets (e.g. Marcy et al. 1999). Resonances within a gaseous disk likely play a key role in determining planetary eccentricities and inclinations. Much of the previous work, starting with Goldreich & Tremaine (1980), has emphasized the effects of mean-motion disk resonances, which involve frequencies that are comparable to the orbital frequencies of the planets.

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Another potentially important class of resonances occurs where there is a matching of the precession frequency of a test particle (e.g. an asteroid) with the frequency of one of the precessional modes of the planetary system. The frequencies involved are much lower than in the mean-motion case. At such a resonance, the motion of a test particle can be strongly driven by the planets, resulting in a high orbital inclination (for a nodal resonance) or eccentricity (for an apsidal resonance). For example, in the solar system there is an important secular resonance that occurs near 2 AU due to driving involving Jupiter and Saturn. This resonance is believed to be associated with the inner truncation of the asteroid belt (Tisserand 1882).

It was recognized by Ward, Colombo, & Franklin (1976) that these resonances must have swept across portions of the early solar system owing to the effects of the gaseous disk, even if the planets do not migrate. The reason is that even a minimum-mass solar nebula can have an important influence on the relevant precession rates. As the nebula disperses, the precession rates vary, along with the resonance locations (Ward 1981). Regions through which the resonances sweep may be driven into significantly eccentric or inclined orbits, such as are observed among the asteroids today (e.g. Nagasawa, Tanaka, & Ida 2000). The resonances can also have an important bearing on the conditions for planet formation by stirring the planetesimal disk.

However, collective effects can be important within the disk and may even result in the development of density waves or bending waves. Consequently, the response is less localized than would be indicated by test particles, as was recognized by Ward & Hahn (1998) and Tremaine (1998). These investigations concentrated on the collective effects of self-gravity in particle disks.

Secular interactions involving the gas disk of a young planetary system are potentially of importance, since much more mass is contained in the gas disk than in the planetesimal disk. The disk is a fluid body and is capable of wave motions through (at least) compressive, inertial, buoyancy, and self-gravitational forces. It is also capable of dissipating energy through a turbulent effective viscosity or through shocks. It is therefore important to understand the dynamical response of the disk to secular forcing and the implications for the evolution of the planetary orbits.

The purpose of this paper is to investigate the outcome of secular interactions in mutually inclined planet-disk systems. The relevant waves in the disk are long-wavelength bending waves with azimuthal wavenumber $m = 1$. The equations governing such waves in a protostellar disk are fairly well established, while those for long-wavelength eccentric density waves have received less attention (although see Ogilvie 2001). The importance of understanding the global $m = 1$ response of the disk to secular perturbations has been emphasized by Tremaine (1998).

The general outline of the paper is as follows. Sections 2–5 describe the physical model for nodal secular interactions and formulate the normal mode analysis. Sections 6 and 7 explore the limit of sufficiently slow modes, when the disk responds nearly rigidly. Section 8 describes how the effects of mean-motion resonances are included in the normal mode analysis. Sections 9 and 10 describe numerical results for Jupiter interacting with the solar nebula. Section 11 provides a simple scaling analysis based on a nearly rigid tilt model. Section 12 discusses the relationship of

the current approach with that of earlier work involving WKB theory. Section 13 describes the analysis of a close-orbiting planet in the central hole of a disk. Section 14 discusses Jupiter and Saturn interacting with the solar nebula at the ν_{16} secular resonance.

2. Modeling secular interactions

Planets experience secular gravitational interactions on time-scales much longer than their orbital periods. On such time-scales the planets may be considered as continuous rings representing their average mass density. These rings are in general elliptical and mutually inclined. Gravitational interactions between the rings lead to apsidal and nodal precession of the orbits.

In the case of small eccentricities and inclinations, the secular evolution of a system of n planets can be described in terms of normal modes of a linear dynamical system, n modes for eccentricity and n for inclination (e.g. Murray & Dermott 1999). The eigenvector of a mode describes the relative distribution of eccentricity or inclination among the planetary orbits, while the eigenfrequency is the rate at which the pattern precesses.

In typical situations involving mean-motion resonances (Lindblad or vertical resonances), if a particle resonance is located at a certain radius in a fluid disk, the disk responds by launching a wave that carries energy and angular momentum away from the resonance. Such waves are usually assumed implicitly to be damped through some dissipative process before reaching an edge of the disk from which they might otherwise reflect. The gravitational torque between the disk and the perturber can then be calculated from a standard formula evaluated at the location of the resonance (Goldreich & Tremaine 1979).

However, in the case of a secular resonance, the frequency may be so small that the wavelength would be comparable to (or may even exceed) the size of the disk. In that case the global response of the disk must be computed, including explicit dissipation and the correct boundary conditions. The exact location of the resonance then ceases to have great importance and the new possibility of a global secular resonance arises.

We focus on the case of inclination, so that the relevant waves in the disk are long-wavelength bending waves with azimuthal wavenumber $m = 1$. As noted above, the equations governing such waves in a protostellar disk are fairly well established.

3. Basic equations

Let (r, ϕ, z) be cylindrical polar coordinates with origin at the central object, of mass M_*. We consider a protostellar disk of semi-thickness $H(r)$, for which the turbulent viscosity parameter α satisfies the condition $\alpha \lesssim H/r$. The linearized equations for bending waves in such a disk have been derived in several papers (Papaloizou & Lin 1995; Masset & Tagger 1996; Demianski & Ivanov

1997). We present them in the form (Lubow & Ogilvie 2000)

$$\Sigma r^2\Omega\frac{\partial W}{\partial t}=\frac{1}{r}\frac{\partial\mathcal{G}}{\partial r}+T, \tag{1}$$

$$\frac{\partial\mathcal{G}}{\partial t}+\left(\frac{\kappa^2-\Omega^2}{\Omega^2}\right)\frac{i\Omega}{2}\mathcal{G}+\alpha\Omega\mathcal{G}=\frac{\mathcal{I}r^3\Omega^3}{4}\frac{\partial W}{\partial r}. \tag{2}$$

Here $W(r,t)=l_x+il_y$ is the complex tilt variable, which describes the warped shape of the disk. Essentially, the disk can be thought of as a continuum of concentric circular rings with radii r and unit normal vectors $\boldsymbol{l}(r,t)$. Also $\mathcal{G}(r,t)=G_x+iG_y$ is the complex internal torque variable. In a warped disk there is a horizontal internal torque $2\pi\boldsymbol{G}(r,t)$ that is mediated by horizontal motions that are proportional to the distance z from the midplane. These stresses are responsible for the propagation of bending waves. Similarly $T(r,t)=T_x+iT_y$ is the complex horizontal external torque density acting on the disk. Finally $\Sigma(r)$ is the surface density, $\Omega(r)$ the orbital angular velocity, $\kappa(r)$ the epicyclic frequency and $\mathcal{I}(r)$ the second vertical moment of the density. These are defined by

$$\Sigma=\int\rho\,dz,\qquad\mathcal{I}=\int\rho z^2\,dz, \tag{3}$$

$$r\Omega^2=\left.\frac{\partial\Phi}{\partial r}\right|_{z=0},\qquad\kappa^2=4\Omega^2+2r\Omega\frac{d\Omega}{dr}, \tag{4}$$

where $\Phi(r,z)$ is the (axisymmetric component of the) gravitational potential experienced by the disk.

Consider a thin, uniform circular ring of mass $m_i\ll M_*$ and radius r_i, representing either an annulus of the disk or the time-average of a planetary orbit. Its contribution to the potential in the plane $z=0$ is

$$\Phi_i=-\frac{Gm_i}{2\pi}\int_0^{2\pi}(r^2+r_i^2-2rr_i\cos\phi)^{-1/2}\,d\phi. \tag{5}$$

Therefore the angular velocity and epicyclic frequency experienced by the ring are obtained from summations over all other such rings, in the forms

$$\Omega_i^2=\frac{GM_*}{r_i^3}+\sum_{j\neq i}\frac{2Gm_j}{r_ir_j}\left[K_0(r_i,r_j)-\frac{r_j}{r_i}K_1(r_i,r_j)\right], \tag{6}$$

$$\kappa_i^2=\frac{GM_*}{r_i^3}+\sum_{j\neq i}\frac{2Gm_j}{r_ir_j}\left[K_0(r_i,r_j)-\frac{2r_j}{r_i}K_1(r_i,r_j)\right], \tag{7}$$

where K_0 and K_1 are symmetric kernels with dimensions of inverse length, given by

$$K_m(r_i,r_j)=\frac{r_ir_j}{4\pi}\int_0^{2\pi}(r_i^2+r_j^2-2r_ir_j\cos\phi)^{-3/2}\cos^m\phi\,d\phi. \tag{8}$$

Hereafter we write K_1 simply as K.

Now consider the tilt interaction between two such rings of masses m_i and m_j, radii r_i and r_j, and tilt vectors $\boldsymbol{l}_i$ and $\boldsymbol{l}_j$. For small relative inclinations, the gravitational torque exerted by ring j on ring i is

$$\boldsymbol{T}_{ji} = Gm_i m_j K(r_i, r_j)\, \boldsymbol{l}_i \times \boldsymbol{l}_j. \tag{9}$$

This expression can be obtained by averaging the torque exerted by a point mass on a circular ring (Lubow & Ogilvie 2000) over the orbital motion of the point mass. Similar expressions can be found in studies of galactic warps (e.g. Sparke & Casertano 1988). In the complex notation, for small inclinations from the xy-plane, the total external torque acting on ring i is

$$T_i = \sum_{j \neq i} Gm_i m_j K(r_i, r_j) i(W_j - W_i). \tag{10}$$

The kernels can also be written as

$$K_m(r_i, r_j) = \frac{r_<}{4r_>^2} b_{3/2}^{(m)} \left(\frac{r_<}{r_>} \right), \tag{11}$$

where $r_> = \max(r_i, r_j)$, $r_< = \min(r_i, r_j)$ and $b_\gamma^{(m)}$ is the Laplace coefficient. In practice we evaluate these in terms of elliptic integrals using Carlson's algorithms (Press et al. 1992).

These kernels diverge as $|r_i - r_j| \to 0$. In reality the gravitational interaction remains bounded because of the non-zero vertical thickness of the rings, which we have neglected. To take account of this in an approximate way, we soften the kernels by replacing

$$K_m(r_i, r_j) \mapsto K_m \left(\frac{r_i + r_j - h}{2}, \frac{r_i + r_j + h}{2} \right), \tag{12}$$

whenever $|r_i - r_j| < h$. Here the smoothing length h is to be understood as an approximate measure of the thickness of the rings.

4. Coupled systems of planets and disks

We consider a general system consisting of multiple planets embedded in a gaseous disk. Provided the planets create gaps in the disk, the disk then becomes partitioned into a set of disks. The disks and planets interact through gravity.

We consider planets $i = 1, 2, \ldots, n_\mathrm{p}$ of masses $m_{\mathrm{p}i}$ in circular orbits of radii $r_{\mathrm{p}i}$ and angular velocities $\Omega_{\mathrm{p}i}$. We also consider disks $k = 1, 2, \ldots, n_\mathrm{d}$ of inner radii a_k and outer radii b_k. The inclinations of the planetary orbits are described by the tilt variables $W_{\mathrm{p}i}(t)$. The tilt within a disk is described by $W_{\mathrm{d}k}(r, t)$. We assume throughout that the inclinations are small, so that linear theory applies.

If only secular nodal interactions are considered, the planets can be treated as inclined circular rings interacting with each other and with the disks through the torques described in Section 3.

We return later (Section 8) to the effect of mean-motion resonances on tilt evolution. However, we assume throughout that any evolution of the surface density of the disk, or of the semi-major axes of the planetary orbits, may be neglected.

The secular equations for the planets are then of the form

$$\begin{aligned} m_{\mathrm{p}i} r_{\mathrm{p}i}^2 \Omega_{\mathrm{p}i} \frac{dW_{\mathrm{p}i}}{dt} &= \sum_{j \neq i} G m_{\mathrm{p}i} m_{\mathrm{p}j} K(r_{\mathrm{p}i}, r_{\mathrm{p}j}) i (W_{\mathrm{p}j} - W_{\mathrm{p}i}) \\ &\quad + \sum_k \int_{a_k}^{b_k} G m_{\mathrm{p}i} \Sigma_k K(r, r_{\mathrm{p}i}) i (W_{\mathrm{d}k} - W_{\mathrm{p}i})\, 2\pi r\, dr, \end{aligned} \tag{13}$$

while the equations for the disks are

$$\begin{aligned} \Sigma_k r^2 \Omega \frac{\partial W_{\mathrm{d}k}}{\partial t} &= \frac{1}{r} \frac{\partial \mathcal{G}_k}{\partial r} + \sum_i G m_{\mathrm{p}i} \Sigma_k K(r, r_{\mathrm{p}i}) i (W_{\mathrm{p}i} - W_{\mathrm{d}k}) \\ &\quad + \sum_\ell \int_{a_\ell}^{b_\ell} G \Sigma_k \Sigma'_\ell K(r, r') i (W'_{\mathrm{d}\ell} - W_{\mathrm{d}k})\, 2\pi r'\, dr', \end{aligned} \tag{14}$$

$$\frac{\partial \mathcal{G}_k}{\partial t} + \left(\frac{\kappa^2 - \Omega^2}{\Omega^2} \right) \frac{i\Omega}{2} \mathcal{G}_k + \alpha \Omega \mathcal{G}_k = \frac{\mathcal{I}_k r^3 \Omega^3}{4} \frac{\partial W_{\mathrm{d}k}}{\partial r}, \tag{15}$$

where $\Sigma'_\ell = \Sigma_\ell(r')$ and $W'_{\mathrm{d}\ell} = W_{\mathrm{d}\ell}(r', t)$. The final term in equation (14) represents the self-gravitation of each disk and the gravitational interactions between disks.

The (complex) total horizontal angular momentum of the system,

$$\sum_i m_{\mathrm{p}i} r_{\mathrm{p}i}^2 \Omega_{\mathrm{p}i} W_{\mathrm{p}i} + \sum_k \int_{a_k}^{b_k} \Sigma_k r^2 \Omega W_{\mathrm{d}k}\, 2\pi r\, dr, \tag{16}$$

is exactly conserved if, as we assume, the boundary conditions

$$\mathcal{G}_k(a_k, t) = \mathcal{G}_k(b_k, t) = 0 \tag{17}$$

hold at all times t. There is also a conservation law for the vertical angular momentum, which we derive in Appendix A.

5. Normal modes

We have a set of coupled integro-differential equations that are linear and homogeneous. Solutions may be sought in the form of normal modes,

$$W_{\mathrm{p}i}(t) = \tilde{W}_{\mathrm{p}i}\, e^{i\omega t}, \qquad W_{\mathrm{d}k}(r, t) = \tilde{W}_{\mathrm{d}k}(r)\, e^{i\omega t}, \qquad \mathcal{G}_k(r, t) = \tilde{\mathcal{G}}_k(r)\, e^{i\omega t}, \tag{18}$$

where ω is a complex frequency eigenvalue. The real part of ω is the precession rate of the mode, while the imaginary part is the decay rate. Substituting this and omitting the tildes, we obtain the integral and integro-differential equations

$$\begin{aligned} i\omega m_{\mathrm{p}i} r_{\mathrm{p}i}^2 \Omega_{\mathrm{p}i} W_{\mathrm{p}i} &= \sum_{j\neq i} G m_{\mathrm{p}i} m_{\mathrm{p}j} K(r_{\mathrm{p}i}, r_{\mathrm{p}j}) i(W_{\mathrm{p}j} - W_{\mathrm{p}i}) \\ &\quad + \sum_k \int_{a_k}^{b_k} G m_{\mathrm{p}i} \Sigma_k K(r, r_{\mathrm{p}i}) i(W_{\mathrm{d}k} - W_{\mathrm{p}i})\, 2\pi r\, dr, \end{aligned} \tag{19}$$

$$\begin{aligned} i\omega \Sigma_k r^2 \Omega W_{\mathrm{d}k} &= \frac{1}{r}\frac{d\mathcal{G}_k}{dr} + \sum_i G m_{\mathrm{p}i} \Sigma_k K(r, r_{\mathrm{p}i}) i(W_{\mathrm{p}i} - W_{\mathrm{d}k}) \\ &\quad + \sum_\ell \int_{a_\ell}^{b_\ell} G \Sigma_k \Sigma_\ell' K(r, r') i(W_{\mathrm{d}\ell}' - W_{\mathrm{d}k})\, 2\pi r'\, dr', \end{aligned} \tag{20}$$

$$i\omega \mathcal{G}_k + \left(\frac{\kappa^2 - \Omega^2}{\Omega^2}\right) \frac{i\Omega}{2} \mathcal{G}_k + \alpha \Omega \mathcal{G}_k = \frac{\mathcal{I}_k r^3 \Omega^3}{4} \frac{dW_{\mathrm{d}k}}{dr}, \tag{21}$$

subject to the boundary conditions

$$\mathcal{G}_k(a_k) = \mathcal{G}_k(b_k) = 0. \tag{22}$$

Frequencies Ω and κ are given by equations (6) and (7). All orbits are assumed to be prograde. There is always a trivial rigid-tilt mode with $\omega = 0$, $\mathcal{G}_k = 0$, and $W_{\mathrm{p}i} = W_{\mathrm{d}k} =$ constant.

6. Rigid response

Secular modes usually have very long periods. If a disk is easily able to maintain radial communication (through pressure, viscosity or self-gravitation) on this long time-scale, it will participate in such a mode almost as a rigid body, and W will not vary much across the disk.

Suppose that each disk indeed behaves rigidly ($W_{\mathrm{d}k} =$ constant). Then $\mathcal{G}_k$ can be eliminated by multiplying equation (20) by $2\pi r$, integrating from a_k to b_k, and using the boundary conditions. We then find

$$\omega J_{\mathrm{p}i} W_{\mathrm{p}i} = \sum_{j\neq i} C_{ij}^{\mathrm{pp}} (W_{\mathrm{p}j} - W_{\mathrm{p}i}) + \sum_k C_{ik}^{\mathrm{pd}} (W_{\mathrm{d}k} - W_{\mathrm{p}i}), \tag{23}$$

$$\omega J_{\mathrm{d}k} W_{\mathrm{d}k} = \sum_i C_{ik}^{\mathrm{pd}} (W_{\mathrm{p}i} - W_{\mathrm{d}k}) + \sum_{\ell\neq k} C_{k\ell}^{\mathrm{dd}} (W_{\mathrm{d}\ell} - W_{\mathrm{d}k}), \tag{24}$$

where

$$J_{\mathrm{p}i} = m_{\mathrm{p}i} r_{\mathrm{p}i}^2 \Omega_{\mathrm{p}i} \tag{25}$$

is the angular momentum of planet i, and

$$J_{\mathrm{d}k} = \int_{a_k}^{b_k} \Sigma r^2 \Omega\, 2\pi r\, dr \tag{26}$$

is the total angular momentum of disk k. The coupling coefficients are defined by

$$C_{ij}^{\rm pp} = Gm_{{\rm p}i}m_{{\rm p}j}K(r_{{\rm p}i}, r_{{\rm p}j}), \tag{27}$$

$$C_{ik}^{\rm pd} = \int_{a_k}^{b_k} Gm_{{\rm p}i}\Sigma_k K(r, r_{{\rm p}i})\, 2\pi r\, dr, \tag{28}$$

$$C_{k\ell}^{\rm dd} = \int_{a_\ell}^{b_\ell}\int_{a_k}^{b_k} G\Sigma_k\Sigma'_\ell K(r, r')\, 2\pi r\, dr\, 2\pi r'\, dr', \tag{29}$$

Evidently the disks can be treated formally as additional planets if they behave as rigid bodies. By introducing Greek indices that run from 1 to $n_{\rm p} + n_{\rm d}$, we obtain the algebraic eigenvalue problem

$$\omega J_\beta W_\beta = \sum_{\gamma\neq\beta} C_{\beta\gamma}(W_\gamma - W_\beta). \tag{30}$$

This is formally identical to the multiple-planet problem described by, e.g., Murray & Dermott (1999). Since C is a symmetric matrix, we have

$$\omega\sum_\beta J_\beta|W_\beta|^2 = -\frac{1}{2}\sum_{\beta,\gamma} C_{\beta\gamma}|W_\beta - W_\gamma|^2. \tag{31}$$

Since, further, the coefficients $C_{\beta\gamma}$ are real and positive, the eigenvalues are all real and non-positive. Apart from the trivial rigid-tilt mode, all modes precess retrogradely without growth or decay.

Suppose the system contains only two components, e.g. one planet interacting with a connected disk. The rigid solution satisfies

$$\begin{bmatrix} \omega J_1 + C & -C \\ -C & \omega J_2 + C \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{32}$$

where $C = C_{12}$ is the coupling coefficient between the planet and the disk. The eigenvalues and eigenvectors are

$$\omega = 0, \qquad \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \tag{33}$$

a trivial solution corresponding to a rigid tilt, and

$$\omega = -\frac{C(J_1 + J_2)}{J_1 J_2}, \qquad \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} \propto \begin{bmatrix} J_2 \\ -J_1 \end{bmatrix}, \tag{34}$$

corresponding to a retrogradely precessing mode in which the tilt of each component is inversely proportional to its angular momentum (in order to conserve the total angular momentum).

In a three-component system, e.g. a planet interacting with interior and exterior disks, the rigid solutions satisfy

$$\begin{bmatrix} \omega J_1 + C_{12} + C_{13} & -C_{12} & -C_{13} \\ -C_{12} & \omega J_2 + C_{12} + C_{23} & -C_{23} \\ -C_{13} & -C_{23} & \omega J_3 + C_{13} + C_{23} \end{bmatrix}\begin{bmatrix} W_1 \\ W_2 \\ W_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{35}$$

where subscript 1 refers to the planet, and subscripts 2 and 3 to the inner and outer disks.

In the general case, the eigenvalues and eigenfunctions are algebraically complicated. However, consider the case in which the planet is much less massive than either disk. The first non-trivial mode is the equivalent of equation (34), but involving the two dominant components:

$$\omega \approx -\frac{C_{23}(J_2+J_3)}{J_2J_3}, \qquad \begin{bmatrix} W_2 \\ W_3 \end{bmatrix} \propto \begin{bmatrix} J_3 \\ -J_2 \end{bmatrix}. \tag{36}$$

The planet's tilt is then driven according to

$$W_1 = \frac{C_{12}W_2 + C_{13}W_3}{\omega J_1 + C_{12} + C_{13}}. \tag{37}$$

Secular resonance occurs here if the frequency is such that the denominator vanishes.

Secular resonance of this type could also occur, for example, in a system of two massive planets and one low-mass disk. In this case the disk would be responding globally as a rigid body. This is quite different from the situation in which a secular particle resonance is located somewhere within a disk.

The second non-trivial mode has

$$\omega \approx -\frac{(C_{12}+C_{13})}{J_1}, \qquad |W_2|, |W_3| \ll |W_1|. \tag{38}$$

Here only the planet's orbit is significantly tilted, and precesses at a rate determined by the two disks.

7. Nearly rigid response

The internal torque $\mathcal{G}_k$ required for a disk to respond rigidly can be determined from equation (20). In general, equation (21) will then indicate that a slight warping is in fact required. The rigid solution can be understood as the leading term in an expansion of the solution in powers of a small parameter. If $\epsilon = H/r$ is the angular semi-thickness of the disk, and $c_{\mathrm{s}} \sim H\Omega$ the isothermal sound speed, the small parameter here is a characteristic value of $(\omega/\epsilon\Omega)^2$, or $(\omega r/c_{\mathrm{s}})^2$. This parameter is small when the disk can communicate effectively, i.e. when the time taken for a bending wave to propagate across the disk is less than the wave period.

Now suppose the disk behaves *nearly* rigidly, so that we may pose an expansion in powers of this parameter, of the form

$$\omega = \omega^{(0)} + \omega^{(1)} + \cdots, \tag{39}$$

$$W_{\mathrm{p}i} = W_{\mathrm{p}i}^{(0)} + W_{\mathrm{p}i}^{(1)} + \cdots, \tag{40}$$

$$W_{\mathrm{d}k} = W_{\mathrm{d}k}^{(0)} + W_{\mathrm{d}k}^{(1)}(r) + \cdots, \tag{41}$$

$$\mathcal{G}_k = \mathcal{G}_k^{(0)}(r) + \mathcal{G}_k^{(1)}(r) + \cdots. \tag{42}$$

At leading order we obtain the rigid solution described in Section 6, denoted by the superscript (0). At the next order we find

$$i\omega^{(1)}J_{\mathrm{p}i}W_{\mathrm{p}i}^{(0)}+i\omega^{(0)}J_{\mathrm{p}i}W_{\mathrm{p}i}^{(1)}=\sum_{j\neq i}Gm_{\mathrm{p}i}m_{\mathrm{p}j}K(r_{\mathrm{p}i},r_{\mathrm{p}j})i(W_{\mathrm{p}j}^{(1)}-W_{\mathrm{p}i}^{(1)})$$

$$+\sum_k\int_{a_k}^{b_k}Gm_{\mathrm{p}i}\Sigma_kK(r,r_{\mathrm{p}i})i(W_{\mathrm{d}k}^{(1)}-W_{\mathrm{p}i}^{(1)})\,2\pi r\,dr, \tag{43}$$

$$i\omega^{(1)}\Sigma_kr^2\Omega W_{\mathrm{d}k}^{(0)}+i\omega^{(0)}\Sigma_kr^2\Omega W_{\mathrm{d}k}^{(1)}=\frac{1}{r}\frac{d\mathcal{G}_k^{(1)}}{dr}+\sum_iGm_{\mathrm{p}i}\Sigma_kK(r,r_{\mathrm{p}i})i(W_{\mathrm{p}i}^{(1)}-W_{\mathrm{d}k}^{(1)})$$

$$+\sum_\ell\int_{a_\ell}^{b_\ell}G\Sigma_k\Sigma_\ell'K(r,r')i(W_{\mathrm{d}\ell}'^{(1)}-W_{\mathrm{d}k}^{(1)})\,2\pi r'\,dr', \tag{44}$$

$$i\omega^{(0)}\mathcal{G}_k^{(0)}+\left(\frac{\kappa^2-\Omega^2}{\Omega^2}\right)\frac{i\Omega}{2}\mathcal{G}_k^{(0)}+\alpha\Omega\mathcal{G}_k^{(0)}=\frac{\mathcal{I}_kr^3\Omega^3}{4}\frac{dW_{\mathrm{d}k}^{(1)}}{dr}, \tag{45}$$

subject to the boundary conditions

$$\mathcal{G}_k^{(1)}(a_k)=\mathcal{G}_k^{(1)}(b_k)=0. \tag{46}$$

After some manipulations we may eliminate $\mathcal{G}_k^{(1)}$ by integration and obtain

$$\omega^{(1)}\sum_\beta J_\beta|W_\beta^{(0)}|^2=\sum_k\int_{a_k}^{b_k}\left(\frac{4}{\mathcal{I}_kr^4\Omega^3}\right)\left[-\omega^{(0)}-\left(\frac{\kappa^2-\Omega^2}{\Omega^2}\right)\frac{\Omega}{2}+i\alpha\Omega\right]|\mathcal{G}_k^{(0)}|^2\,2\pi r\,dr. \tag{47}$$

Since $\mathcal{G}_k^{(0)}$ does not depend on α but is proportional to Σ, this shows that nearly rigid modes are damped at a rate $\mathrm{Im}(\omega^{(1)})$ proportional to α/ϵ^2 (for a disk and planetary system of given dimensions). The imaginary part of equation (47) is a special limit of the conservation law derived in Appendix A.

It is important to note that the self-gravitational term in equation (20) has no effect on the results we have derived. The reason for this is that, if a disk behaves nearly rigidly, *the self-precession of the disk is negligibly small* even if the mass of the disk exceeds that of the planets.

8. Mean-motion resonances

So far we have treated only secular interactions between the planets and disks. However, a planet also interacts with a disk through mean-motion resonances (Lindblad resonances and vertical resonances). The launching of waves at these locations exerts a torque on the planet. If the planet's orbit is inclined with respect to the resonant annulus of the disk, this inclination will evolve in time.

Borderies, Goldreich, & Tremaine (1984, hereafter BGT) calculated the rate of change of inclination of a satellite due to Lindblad and vertical resonances. They showed that the inclination

typically grows exponentially at a rate that depends on the strengths of the resonances present. They assumed implicitly that the launched waves are damped before reaching an edge of the disk from which they might otherwise reflect. In the case of a gas disk, such waves can damp by means of viscous dissipation, radiative damping (Cassen & Woolum 1996), or wave channeling (Ogilvie & Lubow 1999). For narrow gaseous rings between planets, wave damping may well not take place. Instead, reflection from the ring edges may occur, which could greatly reduce the effects of the mean-motion resonances. For simplicity, we will also assume that the waves are damped very close to the resonances and transfer their angular momentum to the disk there.

To incorporate the effect of mean-motion resonances, we modify our equations as follows. (For notational simplicity we here consider only a single planet and a single disk.) The formulae of BGT can be interpreted as giving the horizontal component of the resonant torque between the planet and the disk. We use this to add terms to the planet's angular momentum equation, in the form

$$m_{\rm p}r_{\rm p}^2\Omega_{\rm p}\frac{dW_{\rm p}}{dt} = \cdots + 2\pi\sum_j \frac{Gm_{\rm p}^2}{M_*}s_{{\rm r}j}\Sigma r(W_{\rm p}-W)\Big|_{r=r_{{\rm r}j}}, \tag{48}$$

where $r_{{\rm r}j}$ is the radius of the jth resonance, and $s_{{\rm r}j}$ is the dimensionless strength of the resonance. Similarly, we modify the disk's angular momentum equation to

$$\Sigma r^2\Omega\frac{\partial W}{\partial t} = \cdots + \sum_j \frac{Gm_{\rm p}^2}{M_*}s_{{\rm r}j}\Sigma(W-W_{\rm p})\,\delta(r-r_{{\rm r}j}). \tag{49}$$

As noted above, this assumes that the waves transfer their angular momentum to the disk very close to the resonances. Provided that the disk is not very cold, the precise location of the wave damping is unimportant because the torque is then communicated through the disk over a large radial extent.

The radii and strengths of the resonances are as follows (BGT); here we have neglected the shifts of the resonant radii caused by precession. Inner vertical resonance ($m \geq 2$):

$$x = \frac{r_{\rm r}}{r_{\rm p}} = \left(\frac{m-1}{m+1}\right)^{2/3}, \qquad s_{\rm r} = \frac{\pi x^4}{24(m-1)}\left[b_{3/2}^{(m)}(x)\right]^2. \tag{50}$$

Inner Lindblad resonance ($m \geq 2$):

$$x = \frac{r_{\rm r}}{r_{\rm p}} = \left(\frac{m-1}{m}\right)^{2/3}, \qquad s_{\rm r} = -\frac{m\pi x^2}{12(m-1)}\left[\left(2m + x\frac{d}{dx}\right)b_{1/2}^{(m)}(x)\right]^2. \tag{51}$$

Outer vertical resonance ($m \geq 2$):

$$x = \frac{r_{\rm p}}{r_{\rm r}} = \left(\frac{m-1}{m+1}\right)^{2/3}, \qquad s_{\rm r} = \frac{\pi x^2}{24(m+1)}\left[b_{3/2}^{(m)}(x)\right]^2. \tag{52}$$

Outer Lindblad resonance ($m \geq 1$):

$$x = \frac{r_{\rm p}}{r_{\rm r}} = \left(\frac{m}{m+1}\right)^{2/3}, \qquad s_{\rm r} = \frac{m\pi}{12(m+1)}\left[\left(2m + 1 + x\frac{d}{dx}\right)b_{1/2}^{(m)}(x)\right]^2. \tag{53}$$

These quantities are listed in Tables 1 and 2 for all resonances satisfying $|r_\mathrm{r} - r_\mathrm{p}| > 0.1 r_\mathrm{p}$. Note that the inner Lindblad resonances act in the opposite sense to the other resonances.

Although equation (49) formally requires the disk tilt W to have a cusp at each resonance, these cusps will be very weak in a disk with good radial communication. For a disk that responds nearly rigidly, the effect of the mean-motion resonances is to add a small imaginary part to the planet-disk coupling coefficient C^pd:

$$C^\mathrm{pd} \mapsto C^\mathrm{pd} + 2\pi i \sum_j \frac{G m_\mathrm{p}^2}{M_*} s_{\mathrm{r}j} \Sigma_{\mathrm{r}j} r_{\mathrm{r}j}. \tag{54}$$

The sum over resonances is always positive for an exterior disk but usually negative for an interior disk. According to equation (31), positive sums of this kind tend to cause growth of the non-trivial normal modes. This must compete, however, with the viscous decay implied by equation (47).

9. Model for disk and numerical method

We adopt a simplified model for the solar nebula. The disk model is similar to that described in Lubow & Ogilvie (2000). It is characterized by an angular semi-thickness $H/r = \epsilon =$ constant, and a surface density $\Sigma = \Sigma_0 f r^{-3/2}$, where Σ_0 is a constant and $f(r)$ a function that is close to unity in most of the disk, but tapers linearly to zero at the edges (including gaps around planets). The width of the tapers is equal to the local value of H. The vertical structure is that of a polytrope of index n, which gives $\mathcal{I} = \Sigma H^2/(2n+3)$ (as defined in eq. [3]) if H is the true semi-thickness of the polytrope. The basic parameters of the model are then ϵ, α, and n, together with the dimensions and mass of the disk. We normalize the surface density by quoting a nominal disk mass,

$$m_\mathrm{disk} = \int_0^{r_\mathrm{out}} \Sigma_0 r^{-3/2}\, 2\pi r\, dr = 4\pi \Sigma_0 r_\mathrm{out}^{1/2}, \tag{55}$$

where r_out is the outermost radius. The true disk mass is slightly less than m_disk owing to the various edges.

We consider the nebula in the presence of a Jupiter-like planet. As standard parameters, we adopt $m_\mathrm{p} = 0.001 M_*$, $m_\mathrm{disk} = 0.01 M_*$, $\epsilon = 0.075$, $\alpha = 0.005$, and $n = 3/2$. The planet creates a gap in the nebula, leading to two separate disks whose parameters are taken to be $a_1 = 0.1 r_\mathrm{p}$, $b_1 = (1-g) r_\mathrm{p}$, $a_2 = (1+g) r_\mathrm{p}$, $b_2 = 20 r_\mathrm{p}$. The standard relative half-width of the gap is taken to be $g = 0.2$. This situation may be compared with Jupiter in a solar nebula extending out to about 100 AU. The effect of varying the important parameters will be considered below.

Numerical solutions to the equations of Section 5 are obtained by discretizing the disk, converting the integrals into sums, and converting derivatives into centered finite differences, with $W_{\mathrm{d}k}$ defined on a set of n_r rings for each disk, and $\mathcal{G}_k$ on the boundaries between neighboring rings. Planet-disk and disk-disk interactions are computed with softened kernels K_m, as defined

by equation (12). The smoothing length $h = \max(H_1, H_2)$ is taken to be the maximum of the semi-thicknesses of the two rings concerned. Equation (49) is implemented by identifying the ring in which each resonance falls, and representing the delta function as $1/\delta r$, where δr is the width of the ring. The problem then reduces to a generalized eigenvalue problem involving a non-Hermitian matrix of dimension $n_{\rm p}+n_{\rm d}(2n_{\rm r}-1)$. This is solved numerically using the Fortran Sun Performance Library routine ZGEGV on a Sun Ultra 10, using $n_{\rm r} = 200$. Numerical results are quoted in units such that $G = M_* = r_{\rm p} = 1$. The modes are normalized such that $W_{\rm p} = 1$.

10. Numerical results for Jupiter

We first consider a disk with standard parameters, except that it extends only to $b_2 = 2$ instead of $b_2 = 20$. [Accordingly, we set $m_{\rm disk} = (0.1)^{1/2}0.01$ in this case to fix the same normalization of surface density as in the standard model. The true mass of the disk is only 0.00179.] This case is easier to understand because the disk is able to maintain good radial communication.

Only two of the non-trivial modes are of interest. The first mode ('mode 1') has a precession frequency of -7.02×10^{-4}. The inner disk has a nearly rigid tilt $W \approx -20$ and the outer disk $W \approx +20$. If the mean-motion resonances are neglected, this mode has a viscous decay rate of 4.0×10^{-7}; when they are included, the mode acquires a net growth rate of 1.0×10^{-6}. [This mode corresponds to equation (36) in rigid disk tilt theory.]

The second mode ('mode 2') has a precession frequency of -1.46×10^{-3}. The inner disk has a nearly rigid tilt $W \approx -0.64$ and the outer disk $W \approx -0.75$. If the mean-motion resonances are neglected, this mode has a viscous decay rate of 7.1×10^{-7}; when they are included, the mode acquires a net growth rate of 6.6×10^{-6}.

The remaining non-trivial modes all have significantly larger precession rates and damping rates (with or without mean-motion resonances). They are the proper bending modes of the disks, modified by gravitational coupling to the other components of the system.

When we proceed to the standard model, the numerical solution becomes significantly more complicated. The outer disk extending to $b_2 = 20$ is not able to maintain good radial communication for typical precession rates. The precession frequencies estimated from the nearly rigid theory are comparable to the frequencies of global bending modes in the outer disk. As a result, the nearly rigid theory is no longer accurate for the outer disk.

To simplify the presentation of results, we focus on three modes that are of greatest interest in that they involve the least warping, and have the smallest precession frequencies and damping rates. 'Mode 1' and 'mode 2' are related to the nearly rigid modes mentioned above, while 'mode 3' derives from a bending mode of the outer disk that interferes with the other two when the outer disk fails to behave rigidly. The structures of the three modes are plotted in Fig. 1. The modes include the effects of mean-motion resonances.

In Figs 2 and 3 we plot the precession rates and decay rates of these three modes for the standard model, as we vary each of the parameters α, ϵ, $m_{\rm disk}$, and g about its standard value. Notice that the variation of decay rate with some parameters is non-monotonic. In particular, the decay rate of mode 1 peaks at $m_{\rm disk} \approx 0.02$.

In Appendix A, we derive an expression for the local decay rate of a warped disk due to dissipation. In Fig. 4 we plot for mode 1 the decay rate, as given by equation (A3). Notice that the outer disk dissipation dominates over the inner disk dissipation. Although the dissipation peaks near the planet, it is broadly distributed throughout the outer disk. In the outer disk, the warp $|rdW/dr|$ is greatest near the disk radial midpoint and is broadly distributed. Quantity $|rdW/dr|$ must vanish at the inner and outer disk edges, due to boundary condition (22). The reason that the peak in Fig. 4 in the outer disk occurs near the planet is that the magnitudes of various base-state quantities, such as Σ and Ω, are largest there.

In summary, an inclined planet gives rise to large-scale warps. The damping effects of secular interactions dominate over tilt excitation by mean-motion resonances in our model system, provided $\alpha \gtrsim 0.001$. For the standard model, the decay time of the longest lived mode, mode 1, is about 2.7×10^5 yr. Although this time-scale is short compared to typical disk lifetimes of several million years, it is comparable to or longer than the expected planetary migration time-scale at this radius (Lin, Bodenheimer, & Richardson 1996; Ward 1997).

11. Simple estimates

We compare the numerical results with the nearly rigid theory of Section 7 for the standard disk model. To do this, we carry out some rough estimates and drop all factors of order unity. Consistent with our standard model, we assume below that the mass of the planet is less than, or comparable to, the masses of the two disks, and also that the angular momentum of the outer disk is greater than, or comparable to, the angular momenta of the planet and inner disk.

The coupling coefficients are dominated by the parts of the disk closest to the planet, where $|r - r_{\rm p}|/r_{\rm p} \sim g$. Here the kernel may be estimated as $K \sim 1/(g^2 r)$. Thus

$$C^{\rm pd} \sim \frac{1}{g} G m_{\rm p} \Sigma r \tag{56}$$

and

$$C^{\rm dd} \sim G \Sigma^2 r^3. \tag{57}$$

Here Σ is a typical surface density near the planet. The precession rate of a mode can be estimated from either equation (36) or equation (38) as

$$\omega \sim -\frac{G\Sigma}{g r \Omega}. \tag{58}$$

We then estimate $\mathcal{G}$ from equation (20) as

$$\mathcal{G} \sim i\omega\Sigma r^4 \Omega W. \tag{59}$$

We apply these estimates to equation (47) to obtain an estimate of the viscous damping rate,

$$\mathrm{Im}(\omega) \sim -\frac{\alpha}{\epsilon^2}\frac{1}{g^2}\frac{(\Sigma r^2)^2}{M_*^2}\Omega. \tag{60}$$

When mean-motion resonances are taken into account, the planet-disk coupling coefficient acquires a small imaginary part, as in equation (54). The sum over resonances can be estimated using arguments adapted from BGT. For $g \ll 1$, the number of resonances in the disk scales as $m \sim 1/g$. The strengths of the Lindblad resonances, which dominate here, scale as $s_\mathrm{r} \sim m^2$. Thus

$$\mathrm{Im}(C^\mathrm{pd}) \sim \frac{1}{g^3}\frac{G m_\mathrm{p}^2 \Sigma r}{M_*}. \tag{61}$$

¿From equation (36) this would provide a resonant growth rate

$$\mathrm{Im}(\omega) \sim \frac{1}{g^3}\frac{m_\mathrm{p}\Sigma r^2}{M_*^2}\Omega. \tag{62}$$

The ratio of viscous decay to resonant growth can then be estimated as

$$\frac{\mathrm{decay}}{\mathrm{growth}} \sim \left(\frac{\alpha}{\epsilon^2}\right)\left(\frac{g\Sigma r^2}{m_\mathrm{p}}\right). \tag{63}$$

The first factor depends on the viscosity and temperature of the disk. The second is related to the ratio of the disk mass within (say) two gap widths of the planet to the mass of the planet itself. For our standard parameters, these first factor is close to unity, while the second is somewhat less than unity. This favors tilt growth, provided that the disk indeed behaves nearly rigidly.

12. Connection with WKB bending wave theory

Disks subject to forcing by a misaligned companion have been studied in the context of planetary rings, but with strikingly different results. Within Saturn's A ring, tightly wrapped waves are launched from a mean-motion vertical resonance by a misaligned satellite, as described by Shu, Cuzzi, & Lissauer (1983). Furthermore, the torque carried by the waves is independent of α and ϵ, unlike the present case.

As emphasized earlier, the difference lies in the fact that the waves here are of low frequency such that the parameter $\omega r/c_\mathrm{s}$ is typically less than unity for protostellar disks with secular resonances. For the mean-motion vertical resonances in planetary rings, this quantity is much greater than unity.

Nonetheless, a formal connection between the equations used in this paper and those used for planetary rings can be made by considering an artificially cold protostellar disk, such that $\omega r/c_{\rm s}$ is much greater than unity. Furthermore, we consider the case that the disk mass is much greater than the planetary mass. Appendix B outlines the derivation of the dynamical equations in the WKB limit. In that limit we obtain

$$D(r)W - 2\pi i\sigma_k G\Sigma\frac{dW}{dr} + \left(\frac{\mathcal{I}}{2\Sigma}\right)\frac{\Omega^3}{[\omega + (\kappa^2-\Omega^2)/(2\Omega) - i\alpha\Omega]}\frac{d^2W}{dr^2} \approx \frac{2Gm_{\rm p}}{r^2}K(r_{\rm p},r)W_{\rm p}, \quad (64)$$

where $\sigma_k = \pm 1$ with the sign depending on whether the waves are trailing or leading, and

$$D(r) = 2\omega\Omega + \frac{2Gm_{\rm p}}{r^2}K(r_{\rm p},r). \quad (65)$$

For $\omega \ll \Omega$, we can re-express $D(r)$ in the standard form

$$D(r) = \mu'^2 - (\Omega' - \omega)^2, \quad (66)$$

where

$$\mu'^2 = \left.\frac{\partial^2\Phi'}{\partial z^2}\right|_{z=0}, \qquad r\Omega'^2 = \left.\frac{\partial\Phi'}{\partial r}\right|_{z=0}. \quad (67)$$

Potential Φ' is due to the star and axisymmetric contributions of the planet, *but does not include contributions due to the disk.* This equation is identical in form to equation (20) of Shu et al. (1983) for $m = 1$, apart from the pressure term, the third term in the equation (64).

The pressure term arises because the warping of the disk introduces horizontal pressure gradients. The resulting shearing horizontal motions give rise to the hydrodynamic torque $2\pi\boldsymbol{G}$, which is the dominant factor in the propagation of bending waves in protostellar disks. The warping is described by the $n = 0$ mode in the notation Lubow and Pringle (1993). The dispersion relation for adiabatic oscillations of this mode in an exactly Keplerian disk, whose unperturbed state is vertically isothermal, follows from equation (54) of Lubow and Pringle (1993) with dimensionless frequencies $F = (\Omega - \omega)/\Omega \approx 1$ and $\kappa = 1$. In the low-frequency regime ($|\omega| \ll \Omega$), one obtains then that $\omega = \pm kH\Omega/2$, independent of the exponent for adiabatic compression γ. Although the form of the dispersion relation is reminiscent of that for a compressive acoustic mode, this mode is largely incompressible, as suggested by the lack of dependence on γ in the dispersion relation. The same dispersion relation (as was obtained by Papaloizou & Lin 1995) follows from equation (64), with W varying as $\exp(ikr)$, and $\mathcal{I} = (c_{\rm s}/\Omega)^2\Sigma$, and by taking $G = 0$ (ignoring gravity), $\kappa = \Omega$, and $\alpha = 0$.

For this limit where the planet mass is small compared to the disk mass, the resonance condition $D(r) = 0$ becomes

$$\frac{m_{\rm p}}{m_{\rm d}}\frac{K(r,r_{\rm p})}{\langle K\rangle}\left(\frac{r_{\rm p}}{r}\right)^{1/2} = 1 \quad (68)$$

where $\langle K\rangle$ denotes the mass weighted average of K over the disk (see Appendix B for more details). For the standard disk model, the resonance condition is satisfied at radial distances of about 0.7 and 1.3 within the inner and outer disks respectively.

This system contains a secular particle resonance where the precession rate of a particle matches the precession rate of the disk-planet system. A free particle (e.g. an asteroid) will precess due to the gravitational effects of both the disk and the planet. That is, a free particle resonance will occur when $D(r) = 0$ in equation (66), but Φ' now has an additional contribution due to the disk. The particle resonance location is therefore not the same as the disk resonance location. In other words, in locating a particle secular resonance, the gravity of the gas disk must be included. Particles oscillating vertically in a non-oscillating gas disk have an additional vertical restoring force that alters their precession frequency and shifts their resonance site. This is a special property of the $m = 1$ bending modes (tilt modes), which is a consequence of the fact that there is no self-precession in a rigidly tilted disk. Near resonance, a tilt mode essentially behaves rigidly, due to its locally long wavelength.

In the case of a planetesimal (particle) disk, the nature of the pressure and viscous interaction may well differ from that of a gaseous disk. On the other hand, the secular resonance condition given by equation (68) likely applies to a low-mass planet interacting with a cold planetesimal disk.

13. Planet in central hole

Current models for the early evolution of close-orbiting extra-solar planets, such as 51 Peg b (Mayor & Queloz 1995) with an orbital radius of about 0.05 AU, involve the planet migrating into a central hole in the disk (Lin et al. 1996). The planet is envisioned to migrate into the hole by disk tidal forces until its 2:1 outer Lindblad resonance lies just inside the inner edge of the disk. The planet would then remain at this radius, without further migration.

We consider the tilt stability involving both secular interactions and mean-motion resonances of such a planet-disk system, based on the standard model parameters in Section 9. In this case, there is only an outer disk of mass 0.01 M_*, which extends from just outside the planet's 2:1 outer Lindblad resonance $1.31 r_\mathrm{p}$ (0.066 AU) to outer radius 2000 r_p (100 AU). Other disk parameters are the same as in the standard model ($\epsilon = 0.075$, $\alpha = 0.005$, and $n = 3/2$). The only mean-motion resonance present in the disk is the rather weak 1:3 resonance (2.08 r_p), which acts to increase inclination.

Fig. 5 shows the eigenfunction of the lowest mode of the system. The eigenfrequency is $-5.11 \times 10^{-7} + 4.6 \times 10^{-8} i$ in units of Ω_p, corresponding to a decay time-scale of only 3.9×10^4 yr. The inner part of the disk interacts nearly rigidly with the planet, owing to good radial communication $\omega r_\mathrm{p}/c_\mathrm{s} \ll 1$ (see lower panel of Fig. 5). However, the outer part of the disk is not in good communication ($\omega r/c_\mathrm{s}$ is of order unity), resulting in a warp and dissipation.

14. Disk and two planets

We consider here the case of Jupiter and Saturn interacting with the portion of the solar nebula that lies interior to the orbit of Jupiter, using the equations of Section 5. This case is of interest because the classical ν_{16} nodal secular resonance lies within the disk. We ignore the effects of mean-motion resonances.

The rigid tilt model involving three objects (here the two planets plus disk) was considered at the end of Section 6. As is mentioned there, a secular resonance of the two planets with the disk is possible, if the disk is low in mass compared with the planets. If the disk mass is greater than the planet masses, then the planets individually interact with the disk in a manner similar to the case in Section 10.

We have computed the modes of the system numerically. We consider Jupiter (planet 1) and Saturn (planet 2) in circular orbits with the present values of mass ($m_{\rm p1} = 0.0009545$, $m_{\rm p2} = 0.0002858$) and semi-major axis ($r_{\rm p2} = 1.833$). A test particle would experience nodal secular resonances at semi-major axes 0.381 and 2.39, corresponding to 1.98 AU and 12.4 AU (Murray & Dermott 1999). We consider a partially depleted solar nebula interior to Jupiter's orbit. Our standard model has $m_{\rm disk} = 0.001$, $\epsilon = 0.075$, $\alpha = 0.005$, $n = 3/2$, and disk inner and outer radii $a_1 = 0.1$ and $b_1 = 1 - g = 0.8$, respectively, in units of Jupiter's orbital radius. The inner secular resonance for a test particle therefore lies inside the disk.

For these parameters, the numerically determined eigenfrequencies of the two lowest modes are $-2.27 \times 10^{-4} + 6.0 \times 10^{-9} i$ and $-7.66 \times 10^{-4} + 5.1 \times 10^{-7} i$. The response of the disk is very nearly rigid and nothing special happens at the location of the secular particle resonance. This is to be expected, since $\omega r / c_{\rm s} \ll 1$ throughout the disk. The effect of self-gravitation is negligible even though the mass of the disk is comparable to that of the planets.

If a low-mass disk is extremely cold, it is unable to establish radial communication very effectively on secular time-scales. It cannot respond nearly rigidly, but instead launches a wave at the location of the particle resonance. (The rigid response is essentially the limit of a wavelike response when the wavelength becomes large compared to the size of the disk.) In Fig. 6 we show the wavelike response generated at the ν_{16} secular resonance at 2 AU. A very thin disk ($\epsilon = 10^{-5}$) is required to see this effect, and even then the wavelength is not much shorter than the size of the disk (so a WKB or tight-winding treatment of the wave might not be particularly accurate). We set $\alpha = \epsilon = 10^{-5}$ so that the wave would be partially damped on reaching the inner boundary. We also chose a low disk mass, $m_{\rm disk} = 10^{-6}$, so that the disk has a Toomre parameter $Q > 1$. The eigenfrequency for this lowest mode is $-2.35 \times 10^{-4} + 1.6 \times 10^{-8} i$, corresponding to a damping time of about 1.1×10^8 yr. The resonantly launched waves damp inclination faster than those in the warmer disk model, even though the disk mass is much lower.

15. Summary

We have carried out a normal mode analysis of planet-disk systems that includes the effects of secular interactions and mean-motion resonances. The planets and disks interact gravitationally, and the disks communicate internally through gas pressure, self-gravity, and turbulent viscosity. Secular interactions of misaligned planet-disk systems give rise to global effects in the disks, owing mainly to the effects of gas pressure. The low frequencies ω associated with secular modes allow pressure to communicate effectively over radial distances in disks that are comparable to (and somewhat greater than) the planet's orbital radius. Over such distances, disks behave rigidly with little dissipation. On the other hand, over larger distances ($\sim c_{\rm s}/\omega$), which may arise in a continuous disk exterior to the planet's orbit, the radial communication breaks down and large-scale warps occur, along with enhanced dissipation (see Figs 1 and 5). The protostellar disk response to secular interactions typically falls between that of a global nearly rigidly tilted disk (see Section 7) and that of a locally launched wave (see Sections 12 and 14).

Secular interactions act to decrease inclination by means of turbulent dissipation of horizontal shearing motions within the warp for a simple α disk prescription. The dissipation of the warp is peaked somewhat near the planet, but is broadly distributed throughout the disk (Fig 4). Secular interactions are in competition with the effects of mean-motion resonances (BGT), which can act to increase inclination. For standard disk parameters, the secular interactions involving a single planet generally dominate and the inclination decays in time. For a young Jupiter interacting with the solar nebula, tilt decay takes places for $\alpha \gtrsim 0.001$, with a typical alignment time-scale of order 10^5 yr (see Fig. 3). For a close-orbiting planet in the central hole of the disk, the alignment is more rapid. These results suggest that a planet formed within a disk will remain coplanar with the disk, at least as a consequence of disk interactions.

The classical ν_{16} particle secular resonance, due to Jupiter and Saturn, is expected to have resided within the solar nebula. Again, due to pressure effects, this resonance is greatly broadened in the gas disk. Consequently, the nebula responds nearly rigidly, with some decay in inclination. A very cold planetesimal disk might exhibit a genuinely wavelike response (cf. Fig 6).

The analysis in this paper has focused mainly on the effects of a single planet interacting with its interior and exterior disks. For a system of a multiple planets that opens multiple disk gaps, the situation is more complicated because of reflections that occur at the disk edges associated with the gaps. Such gaps can decrease the effective size of the disk region over which a planet can interact, thereby resulting in a weaker warp and weaker dissipation. The hydrodynamic torque cannot propagate across a gap, while the gravitational torque may at least partially do so. It is quite possible that much longer lived noncoplanar modes may result if the outer part of the nebula is almost disconnected in this way.

This work was supported by NASA grants NAG5-4310 and NAG5-10732, the STScI visitor program, and the Institute of Astronomy visitor program. GIO acknowledges the support of Clare

College, Cambridge through a research fellowship and the Royal Society through a University Research Fellowship.

A. Derivation of the local decay rate of bending disturbances

Starting from equations (13)–(15) we obtain the conservation law

$$\frac{d}{dt}(-L_z) = -\sum_k \int_{a_k}^{b_k} \frac{4\alpha|\mathcal{G}_k|^2}{\mathcal{I}_k r^4 \Omega^2} 2\pi r\, dr, \tag{A1}$$

where

$$-L_z = \sum_i \frac{1}{2} m_{\mathrm{p}i} r_{\mathrm{p}i}^2 \Omega_{\mathrm{p}i} |W_{\mathrm{p}i}|^2 + \sum_k \int_{a_k}^{b_k} \left(\frac{1}{2}\Sigma_k r^2 \Omega |W_{\mathrm{d}k}|^2 + \frac{2|\mathcal{G}_k|^2}{\mathcal{I}_k r^4 \Omega^3} \right) 2\pi r\, dr. \tag{A2}$$

The quantity L_z is the vertical angular momentum associated with the bending disturbance. When an initially horizontal ring of matter having orbital angular momentum J is tilted through an angle $\beta = |W|$, the change in the vertical component of angular momentum is $J(\cos\beta - 1) \approx -\frac{1}{2}J|W|^2$ when $|W| \ll 1$. The term proportional to $|\mathcal{G}|^2$ represents an additional angular momentum stored in the bending wave. For a short-wavelength (WKB) bending wave in a non-self-gravitating disk, both terms in the integral for $-L_z$ are equal. For a nearly rigidly tilted disc, the first term in the integral dominates. In the appropriate limit, this first term agrees with the expression for the angular momentum of a bending wave given by Bertin & Mark (1980, eq. [C12]).

In an inviscid system L_z is conserved. When viscosity is present, the positive definite quantity $-L_z$ decays monotonically to zero. The negative angular momentum of the bending disturbance is transferred to the disk through viscosity, and causes accretion. (It is nevertheless consistent that we neglect the time-dependence of Σ in a linear theory, because the induced accretion rate is quadratic in W.)

We now define a local damping rate $\gamma(r)$ by

$$\gamma = \left(-\frac{1}{L_z}\right) \frac{2\alpha|\mathcal{G}|^2}{\mathcal{I} r^4 \Omega^2} 2\pi r \sum_k (b_k - a_k). \tag{A3}$$

This has the property that

$$\frac{\sum_k \int_{a_k}^{b_k} \gamma\, dr}{\sum_k (b_k - a_k)} = -\frac{1}{2}\frac{d}{dt}\ln(-L_z). \tag{A4}$$

In the case of a normal mode, the radial average of γ corresponds to the damping rate $\mathrm{Im}(\omega)$ of the mode.

B. Derivation of the bending wave equations in the WKB limit

In this Appendix we derive equations (64) and (68) of the text. We consider equations (19)–(21) for a single planet interacting with a single disk.

$$\begin{aligned} i\omega m_{\rm p} r_{\rm p}^2 \Omega_{\rm p} W_{\rm p} &= \int G m_{\rm p} \Sigma K(r, r_{\rm p}) i(W - W_{\rm p})\, 2\pi r\, dr, \quad &\text{(B1)} \\ i\omega \Sigma r^2 \Omega W &= \frac{1}{r}\frac{d\mathcal{G}}{dr} + G m_{\rm p} \Sigma K(r, r_{\rm p}) i(W_{\rm p} - W) \\ &\quad + \int G \Sigma \Sigma' K(r, r') i(W' - W)\, 2\pi r'\, dr', \quad &\text{(B2)} \end{aligned}$$

$$i\omega \mathcal{G} + \left(\frac{\kappa^2 - \Omega^2}{\Omega^2}\right)\frac{i\Omega}{2}\mathcal{G} + \alpha\Omega\mathcal{G} = \frac{\mathcal{I} r^3 \Omega^3}{4}\frac{dW}{dr}. \quad \text{(B3)}$$

We now apply the WKB approximation for a cold disk, $\omega r \gg c_{\rm s}$. In this regime we assume

$$H^{-1} \gg \left|\frac{\partial \ln W}{\partial r}\right| \gg r^{-1}. \quad \text{(B4)}$$

The requirement on H^{-1} is used in the derivation of the warp equations (see Papaloizou & Lin 1995; Lubow and Ogilvie 2000). From equation (B3), we obtain

$$\frac{1}{r}\frac{d\mathcal{G}}{dr} \approx -\frac{i}{4}\frac{\mathcal{I} r^2 \Omega^3}{[\omega + (\kappa^2 - \Omega^2)/(2\Omega) - i\alpha\Omega]}\frac{d^2 W}{dr^2}. \quad \text{(B5)}$$

For the self-gravity term, we have that

$$\int G \Sigma \Sigma' K(r, r') i(W' - W)\, 2\pi r'\, dr' \approx \pm \pi G \Sigma^2 r^2 \frac{dW}{dr}. \quad \text{(B6)}$$

To obtain equation (B6), we recognize that self-gravity is dominated by local contributions in the WKB limit and apply the approximation that $K(r, r') \simeq r/[2\pi(r' - r)^2]$ for $r' \simeq r$. The integral is computed by extending r' to the complex plane and integrating over a closed contour. The contour begins along the negative real axis, includes a small semicircle around the singularity at $r' = r$, continues along the real axis, and closes on itself through a large semicircle. The large semicircle resides in either the upper or lower half-plane, depending on whether the waves are trailing or leading. Equation (B6) then follows from the residue theorem. This choice of trailing or leading waves determines the sign of the result. Substituting equations (B5) and (B6) into equation (B2), we obtain equation (64).

Integrating equation (B2) over the disk area and using equation (B1), we have that

$$J_{\rm d}\langle W \rangle = -J_{\rm p} W_{\rm p}, \quad \text{(B7)}$$

where $\langle W \rangle$ denotes the angular momentum weighted average of W. In the limit that $J_{\rm p} \ll J_{\rm d}$, we expect that W on the right hand side of equation (B1) can be ignored after integration. It then

follows from equation (B1) that

$$\omega = -\frac{\int G\Sigma K(r, r_{\rm p}) 2\pi r\, dr}{r_{\rm p}^2 \Omega_{\rm p}} = -\frac{C^{\rm pd}}{J_{\rm p}}. \tag{B8}$$

Applying this equation for ω to the resonance condition $D(r) = 0$, for D defined by equation (65), we obtain the resonance condition (68).

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Table 1. Mean-motion resonances interior to the planet

$r_{\rm r}/r_{\rm p}$	$s_{\rm r}$	Type	m
0.4807	0.0133	IVR	2
0.6300	−1.1780	ILR	2
0.6300	0.0698	IVR	3
0.7114	0.1916	IVR	4
0.7631	−3.7521	ILR	3
0.7631	0.4009	IVR	5
0.7991	0.7200	IVR	6
0.8255	−7.6763	ILR	4
0.8255	1.1716	IVR	7
0.8457	1.7780	IVR	8
0.8618	−12.9485	ILR	5
0.8618	2.5617	IVR	9
0.8748	3.5451	IVR	10
0.8855	−19.5683	ILR	6
0.8855	4.7508	IVR	11
0.8946	6.2012	IVR	12

Table 2. Mean-motion resonances exterior to the planet

$r_\mathrm{r}/r_\mathrm{p}$	s_r	Type	m
2.0800	0.0191	OVR	2
1.5874	1.4925	OLR	1
1.5874	0.0880	OVR	3
1.4057	0.2272	OVR	4
1.3104	4.3077	OLR	2
1.3104	0.4589	OVR	5
1.2515	0.8055	OVR	6
1.2114	8.4664	OLR	3
1.2114	1.2895	OVR	7
1.1824	1.9334	OVR	8
1.1604	13.9710	OLR	4
1.1604	2.7595	OVR	9
1.1431	3.7904	OVR	10
1.1292	20.8221	OLR	5
1.1292	5.0485	OVR	11
1.1178	6.5564	OVR	12
1.1082	29.0200	OLR	6
1.1082	8.3364	OVR	13
1.1001	10.4110	OVR	14

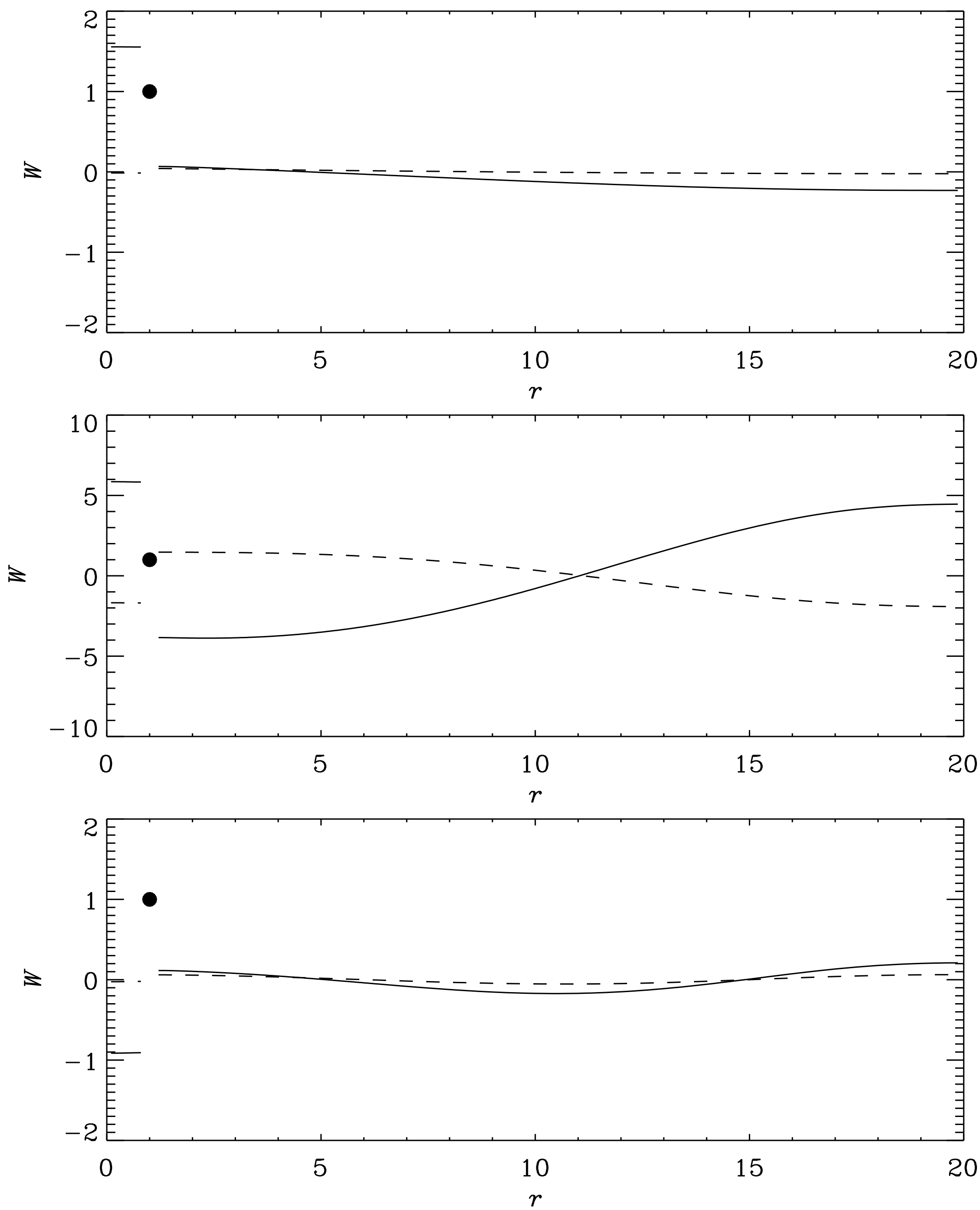

Fig. 1.— Eigenfunctions of the tilt variable W for the three lowest modes, with mode 1 plotted at the top, mode 2 in the middle, and mode 3 at the bottom. The plots are normalized such that the the tilt of the planet (dot) is unity at a radius of unity. The solid curves are for the real part of the eigenfunction, while the dashed curves denote the imaginary part. For the particular disk model adopted (with constant H/r), $W(r)$ is also equal to the vertical displacement from the midplane in units of the local disk scale-height, if the planet resides at one disk scale-height above the midplane.

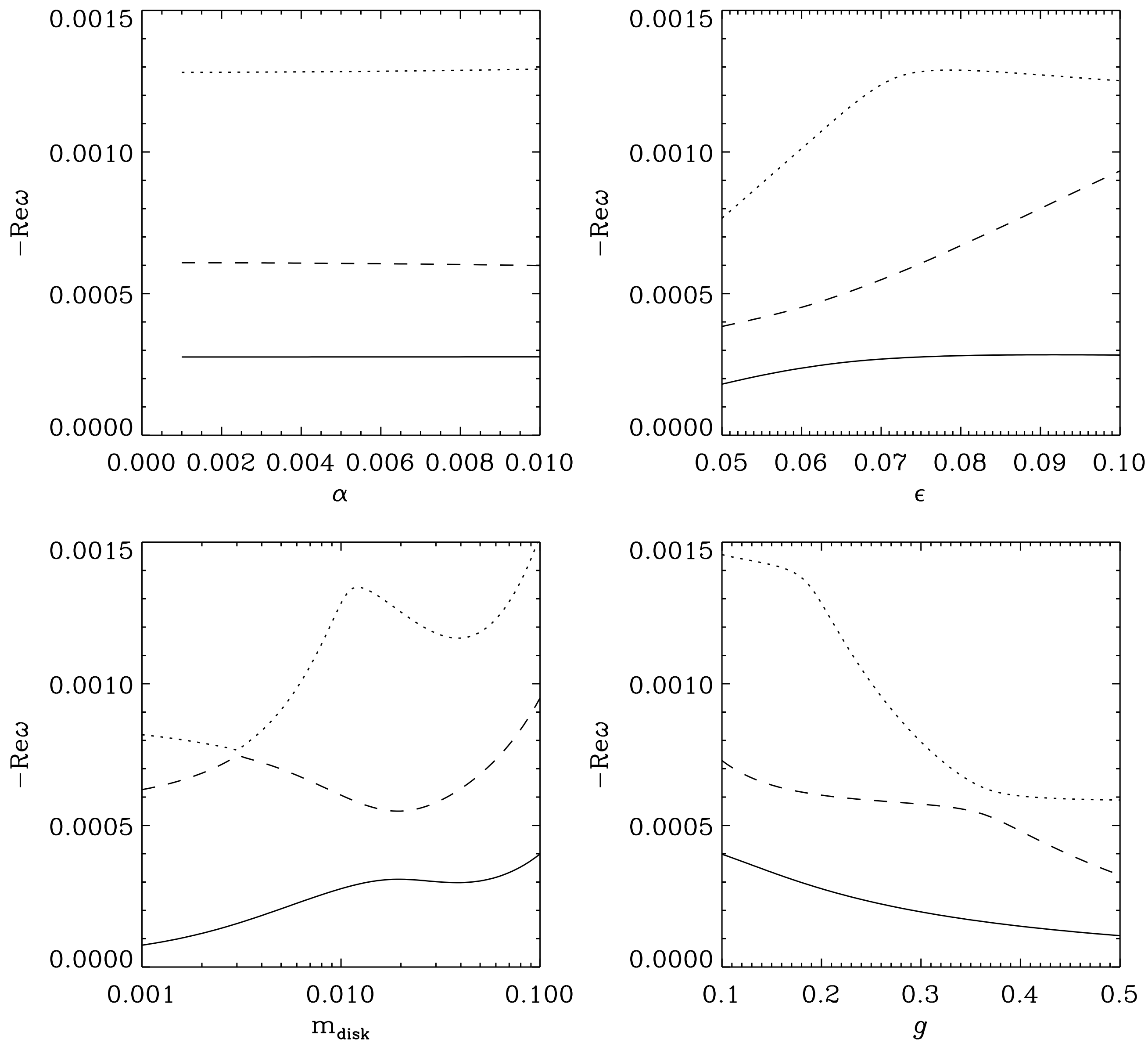

Fig. 2.— Variation of the precession rates of the three lowest modes as the parameters α, ϵ, $m_{\rm disk}$, and g are varied independently about their standard values. The solid, dashed, and dotted lines correspond to modes 1, 2, and 3 referred to in the text.

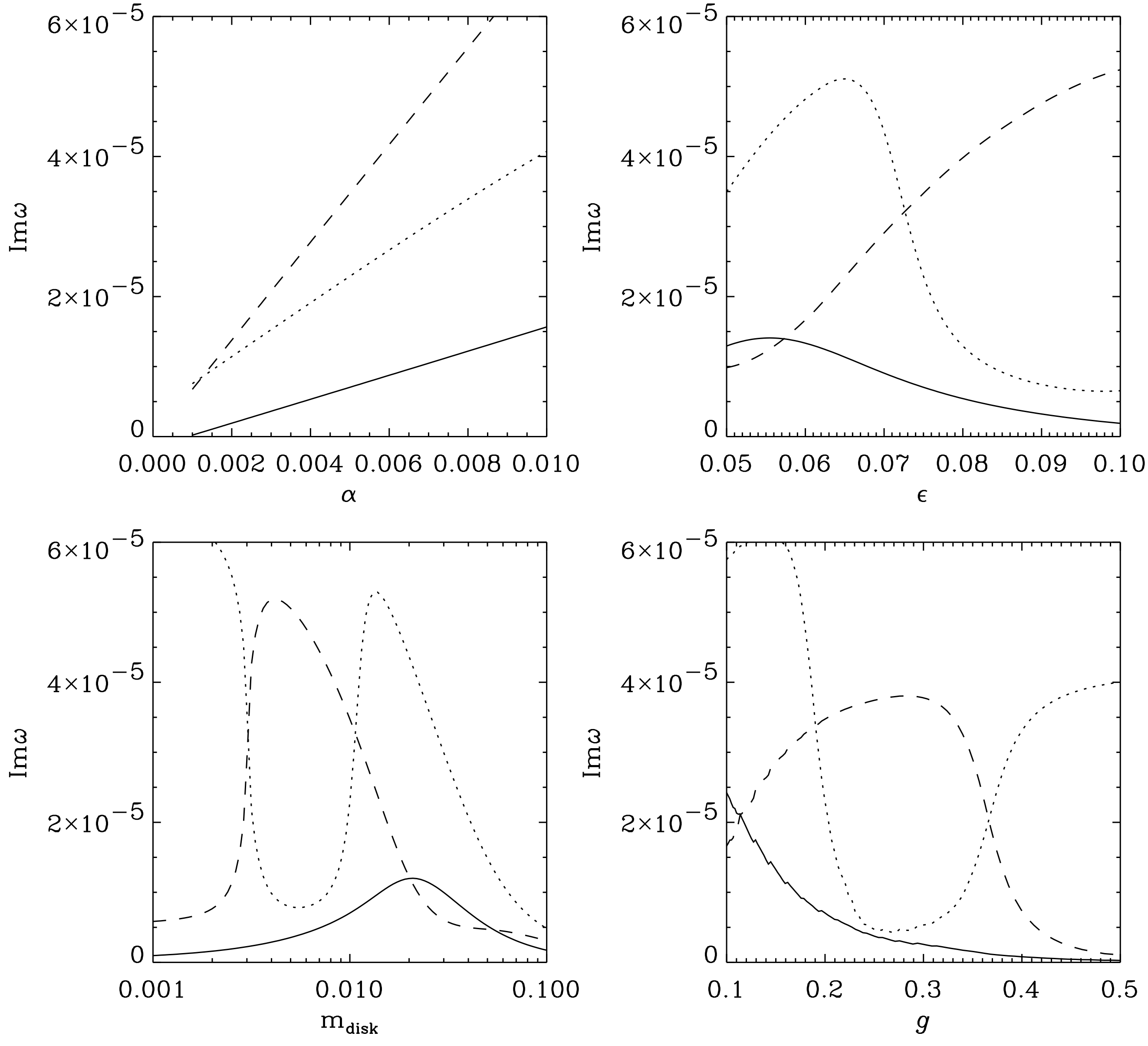

Fig. 3.— Variation of the decay rates. The plot follows the notation of Fig. 2.

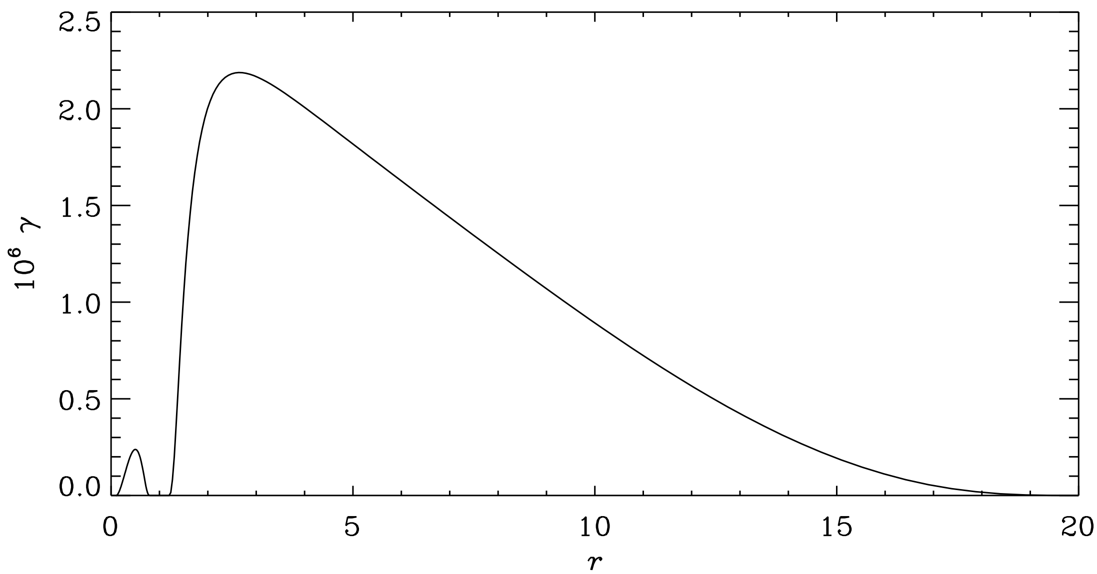

Fig. 4.— Local decay rate of mode 1, as defined in Appendix A, for the standard model.

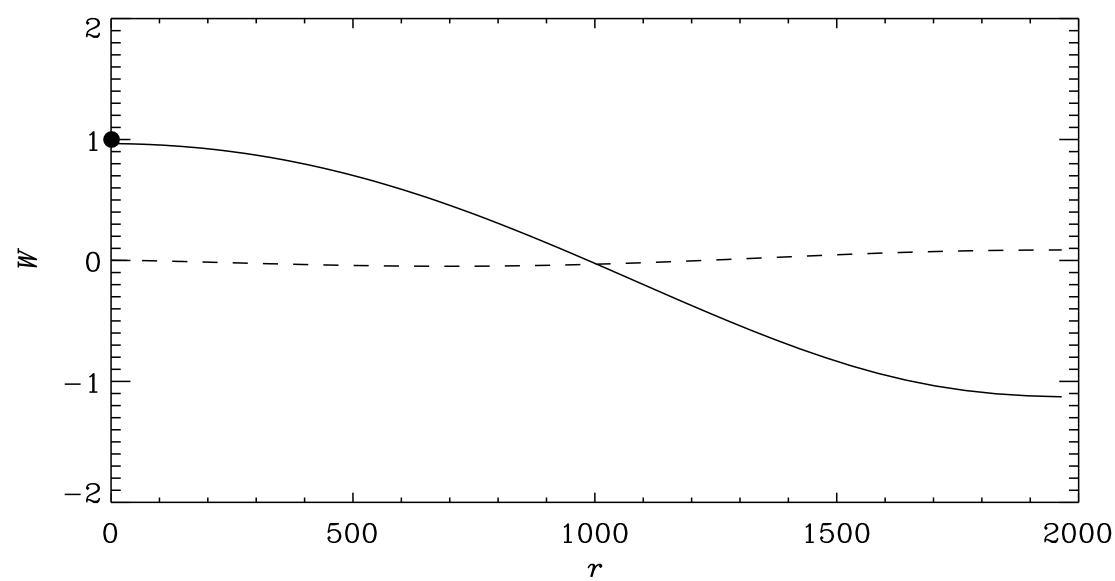

Fig. 5.— Eigenfunction of the tilt variable W for the lowest order mode involving a close-orbiting planet in the central hole of a disk. The plots follow the notation of Fig. 1.

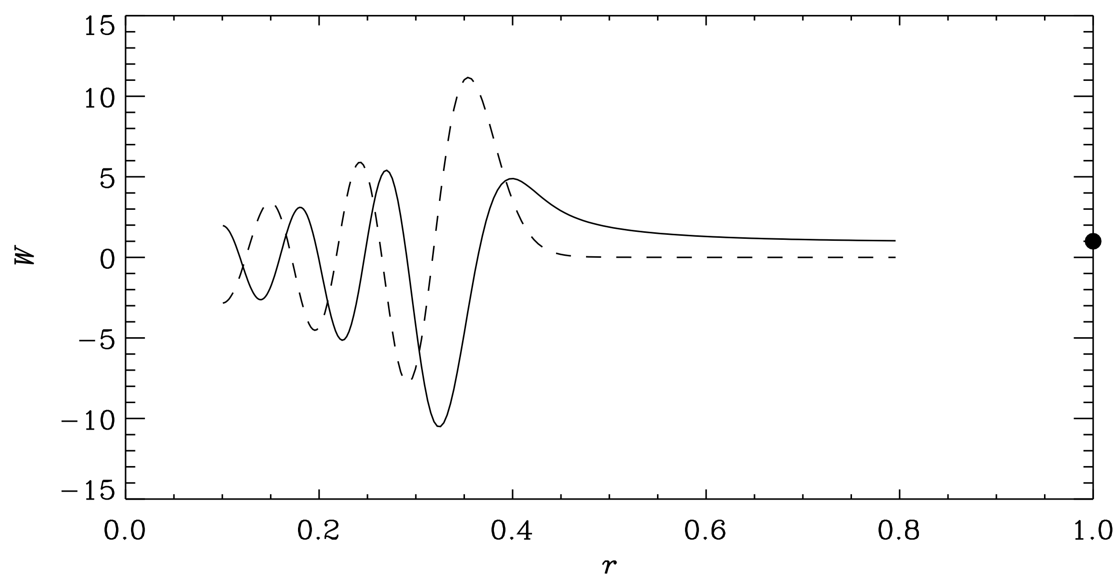

Fig. 6.— Eigenfunction of a secular mode for an artificially cold disk with $\alpha = 10^{-5}$, $\epsilon = 10^{-5}$, and $m_{\rm disk} = 10^{-6}$. Real (solid) and imaginary (dashed) parts of the complex tilt variable are shown. The mode is normalized such that $W_{\rm Jupiter} = 1$, while $W_{\rm Saturn} \approx -2.47$. A wave is launched at the location of the particle secular resonance ($r = 0.381$) due to the Jupiter-Saturn interaction.